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Completeness Property of Generalized Helmholtz Polynomials

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In [2, 3], we have constructed a set of polynomial solutions $\{u_n(x, t; \varepsilon)\}$ for the equation

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t} \tag{1}$$

where ε is a parameter, given by

$$u_n(x, t; \varepsilon) = x^n + n! \sum_{k=1}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!\, k!} \times \sum_{m=0}^{k-1} \frac{(k+m-1)!}{m!(k-m-1)!} t^{k-m} \varepsilon^{2m} \tag{2}$$

which we call generalized Helmholtz polynomials. As $\varepsilon \to 0$, this set of polynomials reduces to the heat polynomials of

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{3}$$

as developed by Rosenbloom and Widder [5].

We have shown [2, p. 101] that every solution $u(x, t)$ of (1) in $t > 0$, continuous in $t \geqslant 0$, whose boundary data $u(x, 0) = \phi(x)$ is an entire function of growth $(1, \sigma)$ with $\sigma < 1/2\varepsilon$, can be expanded as

$$u(x, t) = \sum_{n=0}^{\infty} a_n u_n(x, t; \varepsilon) \tag{4}$$

converging absolutely for $t \geqslant 0$, with $a_n = (1/n!)\, \phi^{(n)}(0)$. For $0 \leqslant \varepsilon < \varepsilon_0$, and (x, t) in any compact subset of the region of convergence, the series (4)

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converges uniformly. This shows the stability of this class of solutions under the singular perturbations of the type exhibited by (1).

Let $t=\varepsilon y$. Equation (1) is transformed into

$$\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}=\frac{1}{\varepsilon}\frac{\partial u}{\partial y} \tag{5}$$

where $\tilde{u}(x, y)=u(x, \varepsilon y)=u(x, t)$. If we let $\tilde{u}(x, y)$ represent the concentration of a gas moving in a given stationary stream with constant velocity $1/\varepsilon$ along the y-axis, Eq. (5) can be interpreted as the diffusion equation of gases in a moving medium with diffusion constant equal to one. The further transformation $\tilde{u}=ve^{y/2\varepsilon}$ leads to

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{1}{4\varepsilon^2}v=0 \tag{6}$$

which is the Helmholtz equation [6, p. 466].

In this note we shall show that $\{u_n(x, t; \varepsilon)\}$ is complete in the maximum norm over any simply connected compact subset in the plane, and provides, after an appropriate transformation, a complete set of solutions for the elliptic equation

$$a\frac{\partial^2\psi}{\partial\xi^2}+2b\frac{\partial^2\psi}{\partial\xi\,\partial\eta}+c\frac{\partial^2\psi}{\partial\eta^2}+d\frac{\partial\psi}{\partial\xi}+e\frac{\partial\psi}{\partial\eta}+f\psi=0 \tag{7}$$

where a, b, c, d, e, and f are constants.

Let $R=\{(x, t) \mid -a<x<a,\ 0<t<T\}$ and let $\bar{R}$ denote the closure of R. Then, we have the following.

THEOREM 1. *Let $u\in C^2(R)\cap C(\bar{R})$ be a solution of (1) in R. Then for $\delta>0$, there exist a positive integer N and constants c_0, $c_1 \cdots c_N$ such that*

$$\max_{\bar{R}}\left|u(x, t)-\sum_{k=0}^{N} c_k u_k(x, t; \varepsilon)\right|<\delta. \tag{8}$$

Proof. The proof is carried out in several steps.

(A) By Weierstrass approximation theorem, there exists a polynomial $P(x, t)$ such that

$$\max_{\bar{R}}|u(x, t)-P(x, t)|<\delta/3. \tag{9}$$

(B) By existence and uniqueness theorem of Dirichlet problem for the elliptic equation (1), we can find a solution $w(x, t)$ of (1) such that $w(x, t) = P(x, t)$ on the boundary of R. Then, by the maximum principle,

$$\max_{\bar{R}} |u(x, t) - w(x, t)| < \delta/3. \tag{10}$$

(C) Let $w(-a, t) = \sum_{m=0}^{M} b_m t^m$, $w(a, t) = \sum_{m=0}^{M} a_m t^m$. We look for a solution $v(x, t)$ of (1) such that $v(-a, t) = w(-a, t)$, $v(a, t) = w(a, t)$. To this end, we set $v(x, t) = \sum_{m=0}^{M} v_m(x)\, t^m$. Substituting v into (1) yields the following recurrence relation for $v_m(x)$:

$$V''_M = 0, \qquad V_M(-a) = b_M, \qquad V_M(a) = a_M;$$

$$V''_{M-1} = M V_M, \qquad V_{M-1}(-a) = b_{M-1}, \qquad V_{M-1}(a) = a_{M-1};$$

$$V''_m = -\varepsilon^2(m+1)(m+2)\, V_{m+2}(x) + (m+1)\, V_{m+1}(x),$$

$$V_m(-a) = b_m, \qquad V_m(a) = a_m, \qquad m = 0, 1, 2, \ldots, M-2.$$

The above boundary value problems always have unique polynomial solutions $V_m(x)$, $m = 0, 1, 2, \ldots, M$.

(D) Let $g(x, t) = w(x, t) - v(x, t)$. Then $g(x, t)$ is a solution of (1) which vanishes at $x = \pm a$, and assumes prescribed polynomial data at $t = 0$ and $t = T$, respectively. The method of separation of variables is applicable and the solution $g(x, t)$ is given by

$$g(x, t) = \sum_{n=1}^{\infty} (\alpha_n e^{P_n t} + \beta_n e^{P'_n t}) \sin \frac{n\pi x}{a}. \tag{11}$$

where

$$P_n = \frac{a - \sqrt{a^2 + 4\varepsilon^2 n^2 \pi^2}}{2a\varepsilon^2}, \qquad P'_n = \frac{a + \sqrt{a^2 + 4\varepsilon^2 n^2 \pi^2}}{2a\varepsilon^2}$$

and α_n, β_n are appropriate constants determined by the boundary data at $t = 0$ and $t = T$. Since the boundary data are analytic, the series (11) converges absolutely and uniformly in $\bar{R}$. Hence, there exists a partial sum $g_n(x, t)$ of $g(x, t)$ such that

$$\max_{\bar{R}} |g(x, t) - g_n(x, t)| < \delta/3.$$

This implies

$$\max_{\bar{R}} |w(x, t) - Q_n(x, t)| < \delta/3 \tag{12}$$

where

$$Q_n(x, t) = v(x, t) + \sum_{j=1}^{n} (\alpha_j e^{P_j t} + \beta_j e^{P'_j t}) \sin \frac{n\pi x}{a}$$

is a solution of (1).

(E) It is easy to see that $Q_n(x, t)$ is continuous in $t \geqslant 0$. Moreover, $Q_n(x, 0)$ is an entire function of growth $(1, \sigma)$ with $\sigma < 1/2\varepsilon$. Hence, we apply Theorem 6.6 in [2, p. 101] and obtain

$$Q_n(x, t) = \sum_{k=0}^{\infty} a_{k,n} u_k(x, t; \varepsilon) \tag{13}$$

where

$$a_{k,n} = \frac{1}{k!} \frac{\partial^k Q_n(x, t)}{\partial x^k} \qquad \text{at } (0, 0).$$

Using bounds for $u_k(x, t; \varepsilon)$

$$|u_k(x, t; \varepsilon)| < 2e^{\rho|x|}[k/e\rho]^k \, e^{(|t|/2\varepsilon^2)(1-\sqrt{1-4\varepsilon^2\rho^2})} \qquad \text{for } \rho < 1/2\varepsilon$$

$$[3, \text{p. } 199] \tag{14}$$

and the growth condition for $a_{k,n}$

$$\limsup_{k \to \infty} |a_{k,n}|^{1/k} (k/e) = \sigma < 1/2\varepsilon \qquad [2, \text{p. } 100]. \tag{15}$$

The series (13) converges uniformly in $\bar{R}$. Hence, there exist a positive integer N and $a_0, \ldots, a_N$ such that

$$\max_{\bar{R}} \left| Q_n(x, t) - \sum_{k=0}^{N} a_{k,n} u_k(x, t; \varepsilon) \right| < \delta/3. \tag{16}$$

Combining (10), (12), and (16), we get (8).

Remarks. (i) The method in this proof is similar to the one used by Colton [1, p. 121] in the case of heat polynomials.

(ii) The theorem is also valid for any closed rectangle in the plane.

(iii) Using singular perturbation technique, Latta [4, p. 458] has proved that for any solution $u(x, t)$ of the Dirichlet problem of (1) in the unit square $S = \{(x, t) | 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant 1\}$,

$$u(x, t) = \sum_{k=0}^{N} a_k(x, t)\, \varepsilon^{2k} + \left(\sum_{k=0}^{N} b_k(x, t)\, \varepsilon^{2k} \right) \exp\left(-\frac{(1-t)}{\varepsilon^2} \right)$$
$$+ O(\varepsilon^{2N+2})$$

as $\varepsilon \to 0$, uniformly in S, where $a_k(x, t)$ and $b_k(x, t)$ can be determined uniquely from the corresponding initial-boundary value problem for the heat equation. It is clear in Theorem 1 that the error term involved for the approximation also depends on ε^{2N+2} as indicated in (2).

To extend Theorem 1 to any simply connected compact subsets in the plane, we rely on the Runge approximation property possessed by solutions of uniformly elliptic equation.

Consider

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \tag{17}$$

where L is a uniformly elliptic operator with analytic coefficients A, B, C, D, E, and F. Solutions of $L[u] = 0$ are said to have the Runge approximation property if, whenever D_1 and D_2 are two bounded simply connected domains, and D_1 is a subset of D_2, any solution in D_1 can be approximated uniformly on compact subsets of D_1 by a sequence of solutions which can be extended as solutions to D_2. It has been shown that solutions of $L[u] = 0$ possess the Runge approximation property by applying Holmgren's uniqueness theorem [1, p. 66–69]. Hence, we have

THEOREM 2. *Let D be a simply connected domain, and let u be a solution of (1) in D. Then, for any $\delta > 0$, and for any simply connected compact subset D_0 of D, there exists a positive integer N and constants $a_0, ..., a_N$ such that*

$$\max_{D_0} \left| u(x, t) - \sum_{k=0}^{N} a_k u_k(x, t; \varepsilon) \right| < \delta.$$

Proof. Choose a rectangle R such that $D \subset R$, and apply Remark (ii) of Theorem 1 and Runge approximation property to solutions of (1) in D_0.

Let $u(x, t)$ be a solution of (1). Define $w(x, y) = u(x, t)\, e^{-t/2\varepsilon^2}$ where $\varepsilon = 1/2\lambda$ and $t = y/2\lambda$. Then $w(x, y)$ is a solution of

$$w_{xx} + w_{yy} - \lambda^2 w = 0. \tag{18}$$

It follows from Theorem 2 that

$$w_n(x, y) = u_n(x, y/2\lambda; 1/2\lambda)\, e^{-\lambda y} \tag{19}$$

is a complete set of solutions of (18) in any simply connected compact subsets in the plane. For the general elliptic equation (7) with constant coefficients, it is well known that there exists a linear transformation

$$x = p_1 \xi + p_2 \eta, \qquad y = p_3 \xi + p_4 \eta, \qquad p_i \text{ are constants} \tag{20}$$

such that Eq. (7) is first carried into

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\alpha\frac{\partial\psi}{\partial x}+\beta\frac{\partial\psi}{\partial y}+\gamma\psi=0 \tag{21}$$

where α, β, and γ are constants.

If $\gamma-\frac{1}{4}(\alpha^2+\beta^2)=-\lambda^2<0$, then (21) is further transformed into (18) where $\psi(x, y)=w(x, y)\exp\{-\frac{1}{2}\alpha x-\frac{1}{2}\beta y\}$.

Since $w_n(x, y)$ is complete for (18), then $\psi_n(\xi, \eta)=w_n(x, y)\exp\{-\frac{1}{2}\alpha x-\frac{1}{2}\beta y\}$ is a complete set for (7) where $\xi=g_1x+g_2y$, $\eta=g_3x+g_4y$ is the inverse for (20). We summarize this in the following theorem:

THEOREM 3. *The set of functions $\{\psi_n(\xi, \eta)\}$ is a complete system of solutions for (7) with respect to uniform convergence in any simply connected compact subsets in the $\xi-\eta$ plane.*

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